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A Hopf bifurcation in a parabolic free boundary problem

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Abstract

The occurrence of a Hopf bifurcation in a free boundary problem for a parabolic partial differential equation is investigated. The bifurcation parameter τ is contained in the equation which describes the evolution of the free boundary. The problem investigated in this paper arises as the singular limit of a system of reaction-diffusion equations with McKean reaction dynamics. Numerical evidence is examined, which shows the emergence of periodic steady states for small positive values of τ. A regularization of the problem is introduced, making it possible to deal with the Heaviside discontinuity in the reaction term, and well-posedness of the free boundary problem is obtained by application of results from the theory of nonlinear evolution equations to the regularized problem. It is then shown that a pair of complex eigenvalues of the linearized problem crosses the imaginary axis as $\tau \to 0$, and the existence of a Hopf bifurcation is proved, using an implicit function theorem argument.

Keywords: Evolution equation; Free boundary problem; Hopf bifurcation; Internal layer solution; Parabolic equation

1. Introduction

In this paper we investigate a Hopf bifurcation in a free boundary problem (more precisely, a free interface problem) for a parabolic partial differential equation. We are dealing with the following problem:

$$\begin{cases} v_t = Dv_{xx} - c^2 v + H(x - s(t)), & \text{for } (x, t) \in \Omega^- \cup \Omega^+, \\ v_x(0, t) = 0 = v_x(1, t), & \text{for } t > 0, \\ v(x, 0) = v_0(x), & \text{for } 0 \leqslant x \leqslant 1, \\ \tau \dfrac{\mathrm{d}s}{\mathrm{d}t} = C(v(s(t), t)), & \text{for } t > 0, \\ s(0) = s_0, \end{cases} \tag{1}$$

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[1] An early version of the results in this paper is contained in this author's dissertation for the Ph.D. in Mathematics at Utah State University.

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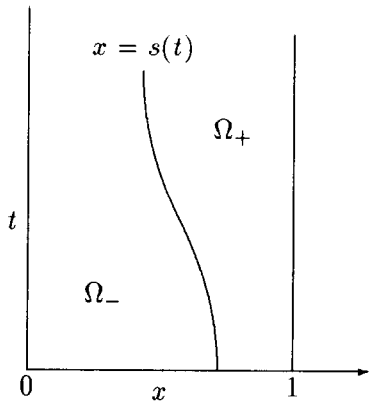

Fig. 1. The (x, t)-domain for problem (1).

where, in addition, $v(x, t)$ and $v_x(x, t)$ are assumed continuous in Ω (this last requirement imposes a kind of boundary condition at the interface). Here $H(y)$ is the Heaviside function, $\Omega = (0, 1) \times (0, \infty)$, $\Omega^- = \{(x, t) \in \Omega: 0 < x < s(t)\}$ and $\Omega^+ = \{(x, t) \in \Omega: s(t) < x < 1\}$ (see Fig. 1).

This problem has its origins in some work of Fife [2,3] on asymptotic analysis of the dynamics of internal layers in reaction-diffusion equations. The free boundary problem is an outgrowth of work done by Mimura, Nishiura and their coworkers [5,8–10]. These authors take as a starting point a system of two reaction-diffusion equations

$$\epsilon\tau u_t = \epsilon^2 u_{xx} + f(u, v), \qquad v_t = Dv_{xx} + g(u, v), \tag{2}$$

depending on two small parameters ϵ and τ. Here u and v measure the levels of two diffusing quantities. The functions u and v are assumed to satisfy Neumann boundary conditions at $x = 0$, 1. The functions f and g are assumed to be of bistable type, i.e., the equation $f = 0$ determines u as a triple-valued function of v and the curves defined by $f = 0$, $g = 0$ have three points of intersection, which determine all of the interactions between u and v. The term bistable refers to the fact that these points of intersection correspond to equilibria of the system (2), two of which are stable, the third unstable. The situation is illustrated in Fig. 2. In the figure the stable equilibria are denoted by $(u_S^{\pm}, v_S^{\pm})$, the unstable equilibrium is denoted by (u_U^0, v_U^0).

When ϵ and τ are chosen to be very small, system (2) models a situation in which the quantity measured by u reacts much faster than that measured by v (τ small), while at the same time u diffuses slower than v (ϵ small). The principal interest in systems like (2) comes from the fact that there exist families of stationary solutions parametrized by ϵ, which approach discontinuous functions of x as $\epsilon \to 0$. When ϵ is small, the stationary solution, being smooth, exhibits an abrupt but continuously differentiable transition at the location of the limiting discontinuity. The transition takes place in an x-interval of length $O(\epsilon)$. An x-interval, in which such an abrupt change takes place, is loosely called a *layer* — a boundary layer when it is adjacent to an endpoint of the interval or an internal layer when it is in the interior of the interval.

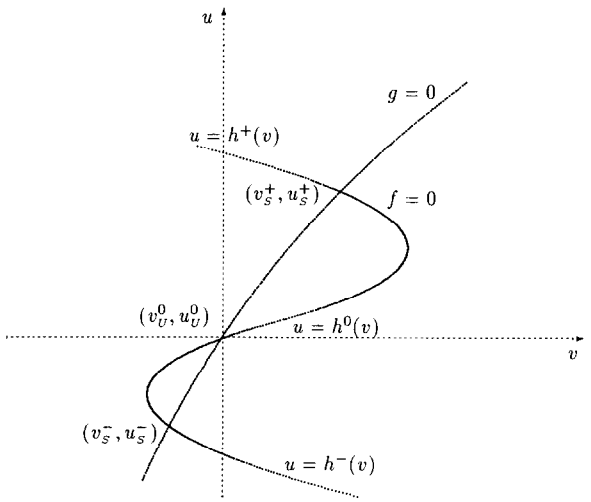

Fig. 2. Null-clines $f = 0$ and $g = 0$ for the bistable system (2).

In 1981, Mimura et al. [8] proved the existence of nontrivial internal layer solutions to the stationary (time-independent) problem associated with (2). The question of the stability of these stationary layer solutions when ϵ is small was later dealt with in a pair of papers; one [9] for the case where τ is large and the solution is asymptotically stable and the second [10] for the case where τ is small and there is a breakdown in the stability of the stationary solutions as τ approaches 0. In the latter paper, a particularly dramatic phenomenon occurs as the stationary solutions lose stability. The loss of stability results from a Hopf bifurcation and produces a kind of periodic oscillation in the location of the internal layers. (The amplitudes of the solutions also undergo a somewhat less pronounced periodic oscillation.) These periodic solutions are called "breathers" or "breathing solutions" because of the nature of the oscillation in the position of the internal layer.

In this paper, we are interested in the singular limit $\epsilon \downarrow 0$ of the system (2). In this case, an analysis of the layer solutions suggests that the layer of width $O(\epsilon)$ converges to an interfacial curve $x = s(t)$ in the (x, t)-space as $\epsilon \downarrow 0$. An analysis of the dynamics of this process has been shown (see, for example, [5,10]) to lead a free boundary problem consisting of the initial-boundary value problem

$$\begin{cases} v_t^{\pm} = Dv_{xx}^{\pm} + g\big(h^{\pm}(v), v\big), & \text{for } (x, t) \in \Omega_{\pm}, \\ v_x^{-}(0, t) = 0 = v_x^{+}(1, t), & \text{for } t > 0, \\ v^{-}(s(t), t) = v^{+}(s(t), t), & \text{for } t > 0, \\ v_x^{-}(s(t), t) = v_x^{+}(s(t), t), & \text{for } t > 0, \\ v^{\pm}(x, 0) = v_0(x), \end{cases} \tag{3}$$

together with an initial-value problem for the interface

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{\tau} C(v(s(t), t)), \quad t > 0, \ s(0) = s_0. \tag{4}$$

Here Ω^-, Ω^+ have the same meaning introduced earlier in problem (1). The function $C(v)$ in (4), which specifies the evolution of the interface $s(t)$, is determined from the first equation in (2) using asymptotic techniques. Details in the derivation of (3) and (4) from (2) can be found in [2,6,10].

In [5], Hilhorst et al. have investigated the well-posedness of problem (3), (4) and they give an analysis of problems which arise in the large diffusion ($D \uparrow \infty$) limit. In the paper, the authors make use of a fixed-point argument to prove results on the existence and uniqueness of solutions. The main result is that the problem has a unique solution on a maximal time interval $[0, T^*]$ and that either $T^* = \infty$ or $T^* < \infty$ and the interfacial curve $s(t)$ hits one of the boundary points at $t = T^*$.

In the present work, we establish the occurrence of a Hopf bifurcation as $\tau \downarrow 0$ in the free boundary problem (3), (4). This free boundary problem comes from the problem (2), which has a Hopf bifurcation, and there is a strong case based on numerical evidence that (3), (4) also experiences a Hopf bifurcation. To see this, consider the case where the reaction terms f and g are of the type investigated by McKean [7], namely

$$f(u, v) = H(u - a) - u - v, \qquad g(u, v) = u - \gamma v,$$

where $H(y)$ is the Heaviside function. For this choice, the velocity of the interface $c(v)$ can be calculated explicitly as

$$c(v) = \frac{2(v + a) - 1}{\sqrt{(1 - v - a)(v + a)}}.$$

The corresponding picture of the null-clines with $\gamma = 1$ and $a = \frac{1}{4}$ is illustrated in Fig. 3.

Using these reaction terms and parameters $D = 1$, $c^2 = 2$, $a = \frac{1}{4}$ and $\gamma = 1$, we have carried out a series of numerical experiments to illustrate the evolution of the free boundary. Graphs of the (x, t)-domain of the solution along with a graph of $s(t)$, for three values of τ, are given

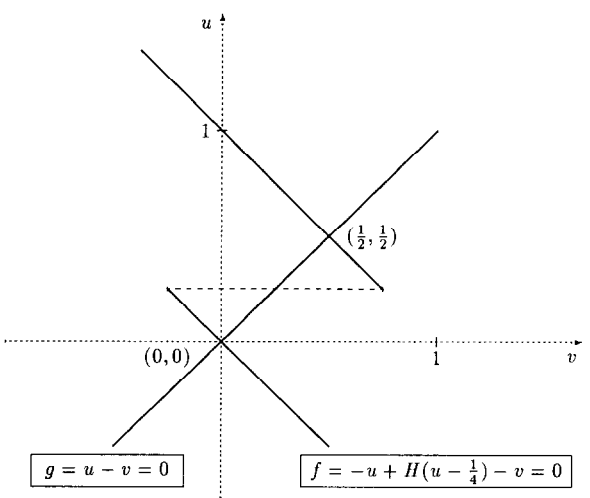

Fig. 3. Reaction terms of McKean [7].

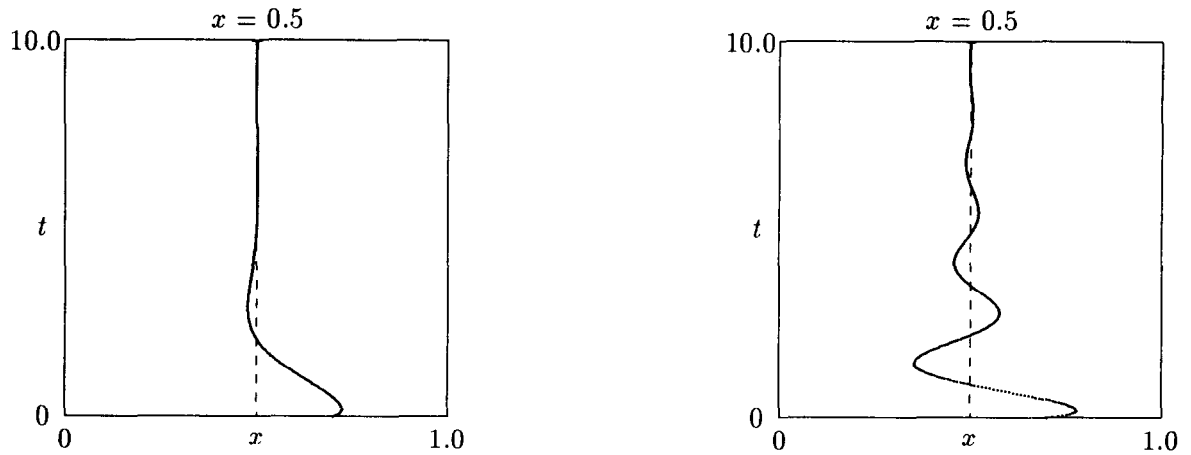

Fig. 4. (left) $\tau = 1.5$; (right) $\tau = 0.5$.

in Figs. 4 and 5. With the parameter values given above, the free boundary problem (1) has a stationary solution $(v^*(x), s^*)$ with $s^* \equiv \frac{1}{2}$ for all τ (illustrated by the dashed line in the center of each of the figures). In each of the simulations, the initial function v_0 was taken to be $v_0(x) = \frac{1}{2}x$.

Fig. 4 (left) shows that for $\tau = 1.5$, the interfacial curve $s(t)$ returns after a short excursion to the equilibrium value $s(t) \equiv \frac{1}{2}$. In Fig. 4 (right) with $\tau = 0.5$, the free boundary $s(t)$ undergoes a damped oscillation about the steady state. In both of these figures, the initial value s_0 of free boundary was taken to be 0.7.

Fig. 5 corresponds to $\tau = 0.17$ and shows the evolution of $s(t)$ toward a periodic steady state. In Fig. 5 (left), starting with the initial condition $s_0 = 0.51$, the curve $s(t)$ spirals outward toward a periodic curve and in Fig. 5 (right) with initial condition $s_0 = 0.7$, the curve $s(t)$ spirals inward toward a similar periodic curve.

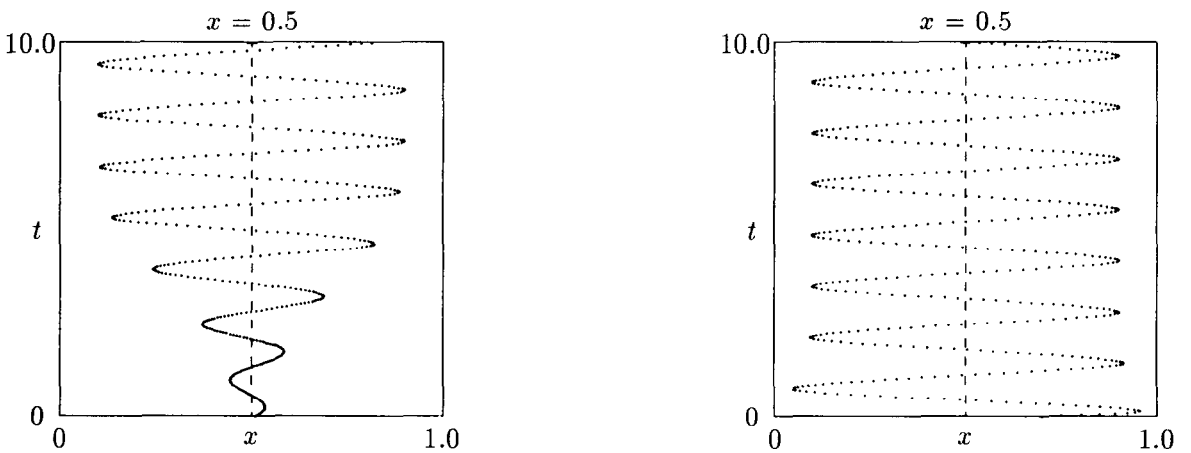

Fig. 5. $\tau = 0.17$.

Based on the results of these numerical approximations, the following picture of the behavior of the free boundary problem as a function of the parameter τ emerges. For large values of τ, the equilibrium solution with $s(t) \equiv 0.5$ is stable and solutions to (1) tend to this equilibrium as $t \to \infty$. As τ decreases, the stability begins to weaken and the convergence of solutions to (1) to the equilibrium solution takes on the form of damped oscillations. At a critical value of τ, the equilibrium solution loses stability and a branch of stable periodic solutions appears — a Hopf bifurcation. These periodic solutions increase in amplitude as τ is further decreased. This suggests a subcritical bifurcation in the sense that there exists a critical τ_c such that the steady state is stable for $\tau > \tau_c$ and unstable for $\tau < \tau_c$, and that τ_c is a bifurcation point for a stable branch of periodic orbits which turns in the direction $\tau < \tau_c$.

In this paper, we will establish the existence of the Hopf bifurcation described above by an application of the implicit function theorem along the lines of the results in [1]. In order to apply the implicit function theorem, we require more regularity of the solution than that obtained in [5]. Our approach to the problem of well-posedness and to the Hopf bifurcation is to write (1) in the form of an abstract evolution equation on a Banach space X which is the product of a function space and an interval of real numbers. Once we have done this, we are able to apply standard results from the theory of nonlinear evolution equations (see, for instance, [4]) to show well-posedness of the problem and, more importantly, give an analysis of the Hopf bifurcation.

Before we proceed to the main results of the paper, we first point out a particular problem which arises in the formulation of (1) as an abstract evolution equation. Briefly stated, the idea is to write (1) in the form

$$\frac{\mathrm{d}(v, s)}{\mathrm{d}t} + A(v, s) = F(v, s), \quad (v, s)(0) = (v_0(\cdot), s_0),$$

of a differential equation in a Banach space X of the form $\tilde{X} = X \times J$, where X is a space of functions and J is a real interval. For the problem (1) this could be done, for instance, by identifying the operator A, represented in matrix form, as

$$A := \begin{pmatrix} -D\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + c^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and the nonlinear operator F by

$$F(v, s) = \begin{pmatrix} F_1(v(\cdot, t), s(t)) \\ F_2(v(\cdot, t), s(t)) \end{pmatrix} := \begin{pmatrix} H(\cdot - s(t)) \\ \dfrac{1}{\tau} C(v(s(t), t)) \end{pmatrix}.$$

The Neumann boundary conditions are incorporated in the definition of the Banach space X.

The difficulty comes from the fact that the nonlinear forcing term $F(v, s)$ contains a Heaviside function in its first component. The combination of this jump discontinuity and the nature of the dependence of v on s in the second component of F makes it impossible to find a function space of the form $X = L_p$, $1 \leqslant p \leqslant \infty$, such that F satisfies a Lipschitz condition on $\tilde{X} \subset X \times \mathbb{R}$.

In Section 2, a change of variables is given which regularizes problem (1) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we give an alternative proof of well-posedness and obtain enough regularity of the solution for an analysis of the bifurcation. In Section 3, we show that as τ decreases, a Hopf bifurcation occurs at a critical value of τ_c. Because of the special nature of the regularization techniques, our results are limited, at present, to the case of McKean reaction dynamics. Also, we will not be able to establish differentiability of the bifurcating solution branch in the bifurcation point, which in turn forbids a straightforward application of a theorem on exchange of stability, though it seems like a generalized version of such a theorem should be applicable here (see numerical results). The final section of the paper contains a brief description of the numerical algorithm used to generate the figures used in the paper.

2. Regularization, existence, uniqueness and dependence on initial conditions

Motivated by the previous section, we now examine a free boundary value problem depending on a parameter $\mu \in \mathbb{R}$ of the form

$$\text{(F)} \qquad \begin{cases} v_t + Av = H(x - s), & x \in (0, 1)\setminus\{s\},\ t > 0, \\ s'(t) = \mu C(v(s(t), t)), & t > 0, \\ v(x, 0) = v_0(x), & s(0) = s_0. \end{cases}$$

Here A is the operator $Av = -v_{xx} + c^2 v$ together with Neumann boundary conditions $v_x(0) = v_x(1) = 0$. Note that by a rescaling of t in (1), we can always achieve that $D = 1$. For the purposes of the results in this section, A can also be any other invertible second-order operator. On the function C, we assume that

$C\colon I \subset_{\text{open}} \to \mathbb{R}$ is continuously differentiable.

For the application of semigroup theory to (F), we choose the space

$X := L_2((0, 1))$, with norm $\|\cdot\|_2$.

Definition 1. We call (v, s) a *solution of* (F), if it satisfies the following natural properties. There exists $T > 0$ such that $v(x, t)$ is defined for $(x, t) \in [0, 1] \times [0, T)$, $s(t) \in (0, 1)$ and $v(s(t), t) \in I$ for $t \in [0, T)$,

(a) $v(\cdot, t) \in C^1([0, 1])$ for $t > 0$ with $v_x(0, t) = v_x(1, t) = 0$,
(b) $s \in C^0([0, T)) \cap C^1((0, T))$ with $s(0) = s_0 \in (0, 1)$,
(c) $(Av)(x, t)$ and $v_t(x, t)$ exist for $x \in (0, 1)\setminus\{s(t)\}$ and $t \in (0, T)$,
(d) $t \mapsto v(\cdot, t) \in C^0([0, T), X)$ with $v(\cdot, 0) = v_0 \in X$, and
(e) v and s solve the differential equation for $t \in (0, T)$ and $x \in (0, 1)\setminus\{s(t)\}$.

As a first step we obtain more regularity for the solution by semigroup methods, considering A as a densely defined operator

$$\begin{cases} A\colon D(A) \subset_{\text{dense}} X \to X, \\ D(A) := \{v \in H^{2,2}((0, 1))\colon v_x(0) = v_x(1) = 0\}. \end{cases}$$

For fixed s satisfying Definition 1, the map $t \mapsto H(\cdot - s(t))$ is locally Hölder-continuous into X on $(0, T)$, so by standard results for parabolic problems (see, e.g., [4]), we obtain from the first equation in (F) that the following regularity holds for v.

Proposition 2. *If (v, s) is a solution of* (F), *then $v(\cdot, t) \in D(A)$ and the map $t \mapsto v(\cdot, t)$ is in $C^0([0, T), X) \cap C^1((0, T), X)$.*

An existence proof for (F) can be obtained along these lines (see [5]), but it is impossible to get differential dependence on initial conditions this way, because the right-hand side $H(\cdot - s)$ is not regular enough, and it is this differentiability that is needed for an application of the Hopf bifurcation theorem.

To overcome this difficulty, we decompose v in (F) into a part u, which is a solution to a more regular problem, and a part g, which is worse but explicitly known in terms of the Green's function G of the operator A.

Proposition 3. *Let G: $[0, 1]^2 \to \mathbb{R}$ be a Green's function of the operator A. Define g: $[0, 1]^2 \to \mathbb{R}$:*

$$g(x, s) := \int_s^1 G(x, y)\, dy = A^{-1}(H(\cdot - s))(x)$$

and γ: $[0, 1] \to \mathbb{R}$:

$$\gamma(s) := g(s, s).$$

Then $g(\cdot, s) \in D(A)$ for all s, $(\partial g/\partial s)(x, s) = -G(x, s)$ is in $H^{1,\infty}((0, 1) \times (0, 1))$ and $\gamma \in C^\infty([0, 1])$.

Proof. Everything follows from the fact that G is in $H^{1,\infty}$ and C^∞ on either $\{x \leqslant y\}$ or $\{x \geqslant y\}$, and that $H(\cdot - s) \in L^2$. □

Using these preliminary observations, we decompose a solution (v, s) of (F) into two parts by defining

$$u(t)(x) := v(x, t) - g(x, s(t)).$$

Then,

$$\begin{aligned} u'(t)(x) &= v_t(x, t) + G(x, s(t))s'(t) \\ &= -(Av)(x, t) + H(x - s(t)) + G(x, s(t))\mu C(v(s(t), t)) \\ &= -(Au(t))(x) + \mu C(u(t)(s(t)) + \gamma(s(t)))G(x, s(t)) \end{aligned}$$

and

$$s'(t) = \mu C(u(t)(s(t)) + \gamma(s(t))).$$

This system can be written as an abstract evolution equation with a nonlinear forcing term f

defined on the set $W := \{(u, s) \in C^1([0, 1]) \times (0, 1)\colon u(s) + \gamma(s) \in I\} \subset_{\text{open}} C^1([0, 1]) \times \mathbb{R}$ as follows:

$$f\colon W \to X \times \mathbb{R}, \quad f(u, s) := f_2(u, s)(f_1(s), 1),$$

where

$$f_1\colon (0, 1) \to X, \ f_1(s)(x) := G(x, s) \quad \text{and} \quad f_2\colon W \to \mathbb{R}, \ f_2(u, s) := C(u(s) + \gamma(s)).$$

We denote the space $X \times \mathbb{R}$ by $\tilde{X}$ and define

$$D(\tilde{A}) := D(A) \times \mathbb{R}, \qquad \tilde{A}\colon D(\tilde{A}) \subset_{\text{dense}} \tilde{X} \to \tilde{X}, \quad \tilde{A}(u, s) := (Au, 0).$$

The initial-value problem for (u, s) can then be written as

$$\text{(R)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}(u, s) + \tilde{A}(u, s) = \mu f(u, s), \qquad (u, s)(0) = (u(0), s(0)) = (u_0, s_0).$$

The advantage of (R) over (F) is that the right-hand side of (R) is one step more regular than that of (F), since it involves $G(x, s)$ instead of $H(x - s)$. More precisely, we can show the following lemma.

Lemma 4. *The functions $f_1 : (0, 1) \to X$, $f_2 : W \to \mathbb{R}$ and $f : W \to \tilde{X}$ are continuously differentiable with derivatives given by*

$$f_1'(s) = \frac{\partial G}{\partial y}(\cdot, s),$$

$$Df_2(u, s)(\hat{u}, \hat{s}) = C'(u(s) + \gamma(s))(u'(s)\hat{s} + \gamma'(s)\hat{s} + \hat{u}(s)),$$

$$Df(u, s)(\hat{u}, \hat{s}) = f_2(u, s)(f_1'(s), 0)\hat{s} + Df_2(u, s)(\hat{u}, \hat{s})(f_1(s), 1).$$

Proof. (a) f_1: The function $(\partial G/\partial y)(x, s)$ exists for $x \neq s$ and is bounded independent of s as a function of x in $L^2((0, 1))$. Moreover, it is continuous almost everywhere in $[0, 1] \times [0, 1]$. Lebesgue's theorem then implies that f_1 is continuously differentiable.

(b) f_2: For f_2 one mainly has to differentiate the function

$$\Gamma\colon C^1[0, 1] \times (0, 1) \to \mathbb{R}, \quad \Gamma(u, s) := u(s).$$

For perturbations $\hat{u}$, $\hat{s}$ of u, s, there exists, by the mean value theorem, a $\theta \in (0, 1)$ such that

$$\begin{aligned}\Gamma(u + \hat{u}, s + \hat{s}) - \Gamma(u, s) - u'(s)\hat{s} - \hat{u}(s) &= u(s + \hat{s}) + \hat{u}(s + \hat{s}) - u(s) - u'(s)\hat{s} - \hat{u}(s) \\ &= \hat{s}(u'(s + \theta\hat{s}) - u'(s)) + \hat{u}'(s + \theta\hat{s})\hat{s} \\ &:= R(\hat{u}, \hat{s})\end{aligned}$$

and

$$\frac{R(\hat{u}, \hat{s})}{\|\hat{u}\|_{C^1} + |\hat{s}|} \to 0, \quad \text{as } |\hat{s}|, \|\hat{u}\|_{C^1} \to 0.$$

As a result, Γ is differentiable with derivative given by

$$D\Gamma(u, s)(\hat{u}, \hat{s}) = u'(s)\hat{s} + \hat{u}(s).$$

Furthermore, since $\|D\Gamma(u_1, s_1) - D\Gamma(u_2, s_2)\| \leqslant \|u_1 - u_2\|_{C^1} + |s_1 - s_2|$, the mapping $(u, s) \mapsto D\Gamma(u, s)$ is continuous.

From the relation $f_2(u, s) = C(\Gamma(u, s) + \gamma(s))$ we obtain that f_2 is continuously differentiable with derivative

$$Df_2(u, s)(\hat{u}, \hat{s}) = C'(\Gamma(u, s) + \gamma(s))(D\Gamma(u, s)(\hat{u}, \hat{s}) + \gamma'(s)\hat{s}),$$

which was to be shown.

Therefore, by the product rule, the derivative of f can be calculated as indicated, and, moreover, Df is continuous. □

We can now apply semigroup theory to (R) using domains of fractional powers $\alpha \in [0, 1]$ of A and $\tilde{A}$:

$$X^\alpha := D(A^\alpha), \qquad \tilde{X}^\alpha := D(\tilde{A}^\alpha), \qquad \tilde{X}^\alpha = X^\alpha \times \mathbb{R}.$$

For this we need to find an $\alpha \in (0, 1)$ such that $X^\alpha \subset C^1([0, 1])$, because then $f: W \cap \tilde{X}^\alpha \to \tilde{X}$ is continuously differentiable [4, Theorem 1.6.1], for example, ensures that this is the case for $\alpha > \frac{3}{4}$. Standard applications of theorems for existence, uniqueness and dependence on initial conditions (cf. [4]) together with the starting regularity of solutions to (F) (Proposition 2) as well as the regularity of the functions g and γ (Proposition 3) then give the following result.

Theorem 5. (i) *For any* $1 > \alpha > \frac{3}{4}$, $(u_0, s_0) \in W \cap \tilde{X}^\alpha$ *and* $\mu \in \mathbb{R}$, *there exists a unique solution*

$$(u, s)(t) = (u, s)(t; u_0, s_0, \mu)$$

of (R). *The solution operator*

$$(u_0, s_0, \mu) \mapsto (u, s)(t; u_0, s_0, \mu)$$

is continuously differentiable from $\tilde{X}^\alpha \times \mathbb{R}$ *into* $\tilde{X}^\alpha$ *for* $t > 0$. *The functions*

$$v(x, t) := u(t)(x) + g(x, s(t))$$

and s *then satisfy* (F) *with* $v(\cdot, 0) \in X^\alpha$, $v(s_0, 0) \in I$.

(ii) *If* (v, s) *is a solution of* (F) *for some* $\mu \in \mathbb{R}$ *with initial condition* $v_0 \in X^\alpha$, $1 > \alpha > \frac{3}{4}$, $s_0 \in (0, 1)$, $v_0(s_0) \in I$, *then* $(u_0, s_0) := (v_0 - g(\cdot, x_0), s_0) \in \tilde{X}^\alpha \cap W$ *and*

$$(v(\cdot, t), s(t)) = (u, s)(t; u_0, s_0, \mu) + (g(\cdot, s(t)), 0),$$

where $(u, s)(t; u_0, s_0, \mu)$ *is the unique solution of* (R).

(iii) *For any* $1 > \alpha > \frac{3}{4}$, $\mu \in \mathbb{R}$, $(v_0, s_0) \in U := \{(v, s) \in X^\alpha \times (0, 1)\colon v(s) \in I\}$ *the problem* (F) *has a unique solution*

$$(v(x, t), s(t)) = (v, s)(x, t; v_0, s_0, \mu).$$

Additionally, the mapping

$$(v_0, s_0, \mu) \mapsto (v, s)(\cdot, t; v_0, s_0, \mu)$$

is continuously differentiable from $X^\alpha \times \mathbb{R}^2$ *into* $X^\alpha \times \mathbb{R}$.

Remark 6. It seems to be difficult to extend this approach for existence and uniqueness to a larger class of initial conditions. If we want the operator f to be locally Lipschitz continuous in

(u, s), then, since it involves the map $\Gamma(u, s) = u(s)$, we necessarily have to take a definition set with $u \in C^{0,1}((0, 1))$. In terms of the problem of finding the right exponent α for $\tilde{X}^\alpha$ there is no difference between C^1 and $C^{0,1}$.

3. Stationary solutions and Hopf bifurcation

We now assume the special forms for A and C which result from the bistable system with the McKean reaction mechanism

$$Au := = -u_{xx} + c^2 u, \quad c > 0,$$

$$C: I \to \mathbb{R}, \quad I = (-a, 1-a),\ 0 < a < 1,\ C(r) := \frac{1}{4} \frac{2a + 2r - 1}{\sqrt{(a+r)(1-a-r)}},$$

which corresponds to choosing $\mu = 4/\tau$ as a new parameter. With this choice, the function $\gamma(s) = \int_s^1 G(s, y)\, dy$ then becomes

$$\gamma(s) = \frac{1}{2c^2}\left(1 - \frac{\sinh(c(2s-1))}{\sinh c}\right),$$

and we have

$$\gamma'(s) < 0, \quad \gamma(0) = \frac{1}{c^2},\ \gamma(1) = 0. \tag{5}$$

The stationary problem, corresponding to (R), is given by

$$Au^* = \mu f_2(u^*, s^*) f_1(s^*), \qquad 0 = \mu f_2(u^*, s^*),$$

for $(u^*, s^*) \in D(\tilde{A}) \cap W$. For $\mu \neq 0$ this system is equivalent to the pair of equations

$$u^* = 0, \qquad C(\gamma(s^*)) = 0. \tag{6}$$

We thus obtain the following proposition.

Proposition 7. *If* $0 < \frac{1}{2} - a < 1/c^2$, *then* (R) *has a unique stationary solution* $(0, s^*)$ *for all* $\mu \neq 0$ *with* $s^* \in (0, 1)$. *The linearization of* f *at* $(0, s^*)$ *is*

$$Df(0, s^*)(\hat{u}, \hat{s}) = (\hat{u}(s^*) + \gamma'(s^*)\hat{s})(f_1(s^*), 1).$$

The pair $(0, s^*)$ *corresponds to a unique steady state* (v^*, s^*) *of* (F) *for* $\mu \neq 0$ *with*

$$v^*(x) = g(x, s^*).$$

Proof. Since $C(r) = 0$ iff $r = \frac{1}{2} - a$, (6) is solvable with $s^* \in (0, 1)$ iff (see (5)) $\gamma(0) > \frac{1}{2} - a > \gamma(1)$, which means $1/c^2 > \frac{1}{2} - a > 0$.

The formula for $Df(0, s^*)$ follows from Lemma 4 and the relation $C'(\frac{1}{2} - a) = 1$. The corresponding steady state (v^*, s^*) for (F) is obtained using Theorem 5. □

We next want to show that there is a Hopf bifurcation from the curve $\mu \mapsto (0, s^*)$ of steady states, and we therefore introduce the following definition.

Definition 8. Under the assumptions of Proposition 7, define (for $1 \geqslant \alpha > \frac{3}{4}$) the operator $B \in L(\tilde{X}^\alpha, \tilde{X})$:

$$B := Df(0, s^*).$$

We then define $(0, s^*, \mu^*)$ to be a *Hopf point* for (R) if and only if there exists an $\epsilon_0 > 0$ and a C^1-curve

$$(-\epsilon_0 + \mu^*, \mu^* + \epsilon_0) \mapsto (\lambda(\mu), \phi(\mu)) \in \mathbb{C} \times \tilde{X}_{\mathbb{C}}$$

($Y_{\mathbb{C}}$ denotes the complexification of the real space Y) of eigendata for $-\tilde{A} + \mu B$ with

(i) $(-\tilde{A} + \mu B)(\phi(\mu)) = \lambda(\mu)\phi(\mu)$, $(-\tilde{A} + \mu B)(\overline{\phi(\mu)}) = \overline{\lambda(\mu)}\,\overline{\phi(\mu)}$;
(ii) $\lambda(\mu^*) = i\beta$ with $\beta > 0$;
(iii) $\operatorname{Re} \lambda \neq 0$ for all $\lambda \in \sigma(-\tilde{A} + \mu^* B) \setminus \{\pm i\beta\}$;
(iv) $\operatorname{Re} \lambda'(\mu^*) \neq 0$ (transversality).

A Hopf point $(0, s^*, \mu^*)$ is the origin of a C^0-curve of initial conditions (u_0, s_0) for nontrivial periodic solutions. This basically follows from a theorem in [1], but the proof requires a little reinvestigation, for the theorem is only stated for C^2-nonlinearities f and then yields a C^1-curve of bifurcating periodic orbits. Since we are unable to meet the C^2 requirement, we indicate briefly how to modify the proof, using an implicit function theorem that only requires differentiability with respect to one part of the arguments.

Theorem 9 (Hopf bifurcation). *Assume* $(0, s', \mu^*)$ *is a Hopf point for* (R). *Then there exists* $\epsilon_1 > 0$ *and a* C^0*-curve*

$$\epsilon \in (-\epsilon_1, \epsilon_1) \mapsto (u_0(\epsilon), s_0(\epsilon), p(\epsilon), \mu(\epsilon)) \in \tilde{X}^\alpha \times \mathbb{R}^+ \times \mathbb{R},$$

such that

$$(u, s)(\cdot; u_0(\epsilon), s_0(\epsilon), \mu(\epsilon))$$

is a periodic solution of (R) *with* (*primitive*) *period* $p(\epsilon)$.

Moreover, $u_0(0) = 0$, $s_0(0) = s^*$, $p(0) = 2\pi/\beta$, $\mu(0) = \mu^*$ *and*

$$\lim_{\epsilon \to 0} \frac{(u_0(\epsilon), s_0(\epsilon) - s^*)}{\epsilon} = \operatorname{Re} \phi(\mu^*).$$

Proof. We follow the argument of [1]. By a rescaling, we can first assume that $\beta = 2\pi$. Then by a second rescaling involving the unknown period p of the bifurcating orbits, we can restrict ourselves to finding one-periodic solutions.

Define $C_1(\tilde{X}^\alpha)$ to be the space of continuous functions from $\mathbb{R}$ to $\tilde{X}^\alpha$ with period 1, and $C_0(\tilde{X}^\alpha)$ as the space of continuous functions $h : [0, 1] \to \tilde{X}^\alpha$ with $h(0) = 0$. Without loss of

generality, we can assume f has been extended to all of $\tilde{X}^\alpha$. We abbreviate $(u, s) \in \tilde{X}^\alpha$ as $\tilde{u}$. Define then $F: C_1(\tilde{X}^\alpha) \times \mathbb{R}^+ \times \mathbb{R} \to C_0(\tilde{X}^\alpha)$ by the equation

$$F(\tilde{u}, p, \mu)(t) := \tilde{u}(t) - e^{-pt\tilde{A}}\tilde{u}(0) - p\mu \int_0^t e^{-p(t-\tau)\tilde{A}} f((0, s^*) + \tilde{u}(\tau))\, d\tau.$$

Solving $F(\tilde{u}, p, \mu) = 0$ is equivalent to finding a p-periodic solution to (R) for the parameter μ.

The kernel of the operator $D_{\tilde{u}}F(0, 1, \mu^*)$ is spanned by $\{\psi_1, \psi_2\}$ with

$$\psi_1(t) = e^{(-\tilde{A}+\mu^* B)t} \operatorname{Re} \phi(\mu^*), \qquad \psi_2(t) = e^{(-\tilde{A}+\mu^* B)t} \operatorname{Im} \phi(\mu^*).$$

If V is chosen such that $C_1(\tilde{X}^\alpha) = \operatorname{span}\{\psi_1, \psi_2\} \oplus V$ and the operator $G: \mathbb{R} \times V \times \mathbb{R}^+ \times \mathbb{R} \to C_0(\tilde{X}^\alpha)$ is defined by

$$G(\epsilon, \tilde{v}, p, \mu) := \begin{cases} \dfrac{1}{\epsilon} F(\epsilon(\psi_1 + \tilde{v}), p, \mu), & \epsilon \neq 0, \\ D_{\tilde{u}} F(0, p, \mu)(\psi_1 + \tilde{v}), & \epsilon = 0, \end{cases}$$

then $G(0, 0, 1, \mu^*) = 0$, and by the same argument as in [1], the transversality condition implies that

$D_{(\tilde{v}, p, \mu)} G(0, 0, 1, \mu^*)$ is an isomorphism.

(Note that $G(0, \cdot, \cdot, \cdot)$ is C^1.) An implicit function theorem which yields C^0-curves of solutions is, e.g., formulated in [11, p.170]. We want to obtain a solution curve parametrized by ϵ, and therefore have to check that

G is continuous, (7)

and that the mapping

$(\epsilon, \tilde{v}, p, \mu) \mapsto D_{(\tilde{v}, p, \mu)} G(\epsilon, \tilde{v}, p, \mu)$ is continuous. (8)

The continuity condition (7) follows from Df being continuous. For (8), we observe that

$$D_{\tilde{v}} G(\epsilon, \tilde{v}, p, \mu)\hat{v} = \begin{cases} D_{\tilde{u}} F(\epsilon(\psi_1 + \tilde{v}), p, \mu)\hat{v}, & \epsilon \neq 0, \\ D_{\tilde{u}} F(0, p, \mu)\hat{v}, & \epsilon = 0, \end{cases}$$

so $D_{\tilde{v}} G$ is continuous since Df is. For D_p, we have

$$D_p G(\epsilon, \tilde{v}, \mu) = \begin{cases} \dfrac{1}{\epsilon} D_p F(\epsilon(\psi_1 + \tilde{v}), p, \mu), & \epsilon \neq 0, \\ D_{\tilde{u}} D_p F(0, p, \mu)(\psi_1 + \tilde{v}), & \epsilon = 0, \end{cases}$$

which is continuous since $D_p D_{\tilde{u}} F = D_{\tilde{u}} D_p F$ is. By a similar argument, $D_\mu G$ is continuous.

As a result, by the implicit function theorem, there is a C^0-curve $\epsilon \mapsto (\tilde{v}(\epsilon), p(\epsilon), \mu(\epsilon))$ such that $G(\epsilon, \tilde{v}(\epsilon), p(\epsilon), \mu(\epsilon)) = 0$ with $\tilde{v}(0) = 0$, $p(0) = 1$ and $\mu(0) = \mu^*$. This translates into a curve $\epsilon \mapsto (u_0(\epsilon), s_0(\epsilon), p(\epsilon), \mu(\epsilon))$ which has the required properties via the definition

$$(u_0(\epsilon), s_0(\epsilon)) := (0, s^*) + \epsilon\psi_1(0) + \epsilon\tilde{v}(\epsilon)(0).$$

Note that the main reason why this proof works is that $D_{\tilde{u}}F$ can be differentiated again with respect to μ and p. □

We next have to check (R) for Hopf points. For this we have to solve the eigenvalue problem

$$-\tilde{A}(u, s) + \mu B(u, s) = \lambda(u, s),$$

which by Proposition 7 is equivalent to

$$(A + \lambda)u = \mu(\gamma'(s^*)s + u(s^*))G(\cdot, s^*), \qquad \lambda s = \mu(\gamma'(s^*) + u(s^*)). \tag{9}$$

As a first result, we obtain that it suffices to find a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (9) with $\beta > 0$ for some μ^* in order for $(0, s^*, \mu^*)$ to be a Hopf point.

Theorem 10. *Assume that for $\mu^* \in \mathbb{R}\setminus\{0\}$ the operator $-\tilde{A} + \mu^* B$ has a unique pair $\{\pm i\beta\}$ of purely imaginary eigenvalues. Then $(0, s^*, \mu^*)$ is a Hopf point for* (R).

Proof. Without loss of generality, let $\beta > 0$, and let ϕ^* be the (normalized) eigenfunction of $-\tilde{A} + \mu^* B$ with eigenvalue $i\beta$. We have to show that $(\phi^*, i\beta)$ can be extended to a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata for $-\tilde{A} + \mu B$ with $\lambda'(\mu^*) \neq 0$.

For this let $\phi^* = (\psi_0, s_0) \in D(A) \times \mathbb{R}$. First, we see that $s_0 \neq 0$, for otherwise, by (9), $(A + i\beta)\psi_0 = i\beta s_0 G(\cdot, s^*) = 0$, which is not possible because A is symmetric. So without loss of generality, let $s_0 = 1$. Then by (9), $E(\psi_0, i\beta, \mu^*) = 0$, where

$$E\colon D(A)_{\mathbb{C}} \times \mathbb{C} \times \mathbb{R} \to X_{\mathbb{C}} \times \mathbb{C},$$

$$E(u, \lambda, \mu) := \big((A + \lambda)u - \mu(\gamma'(s^*) + u(s^*))G(\cdot, s^*),\ \lambda - \mu(\gamma'(s^*) + u(s^*))\big).$$

The equation $E(u, \lambda, \mu) = 0$ is equivalent to λ being an eigenvalue of $-\tilde{A} + \mu B$ with eigenfunction $(u, 1)$. We want to apply the implicit function theorem to E, and therefore have to check that E is C^1 and that

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu_0) \in L\big(D(A)_{\mathbb{C}} \times \mathbb{C},\ X_{\mathbb{C}} \times \mathbb{C}\big) \text{ is an isomorphism.} \tag{10}$$

Now it is easy to see that

$$\begin{cases} D_u E(u, \lambda, \mu)\hat{u} = \big((A + \lambda)\hat{u} - \mu\hat{u}(s^*)G(\cdot, s^*),\ -\mu\hat{u}(s^*)\big), \\ D_\lambda E(u, \lambda, \mu)\hat{\lambda} = \hat{\lambda}(u, 1), \\ D_\mu E(u, \lambda, \mu)\hat{\mu} = -\hat{\mu}(\gamma'(s^*) + u(s^*))(G(\cdot, s^*), 1). \end{cases} \tag{11}$$

so E is C^1. In addition, the mapping

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\hat{u}, \hat{\lambda}) = \big((A + i\beta)\hat{u} - \mu^*\hat{u}(s^*)G(\cdot, s^*) + \hat{\lambda}\psi_0,\ -\mu^*\hat{u}(s^*) + \hat{\lambda}\big)$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{\lambda}) \mapsto \big((A + i\beta)\hat{u}, \hat{\lambda}\big),$$

which is invertible. As a consequence, $D^{(u,\lambda)}E(\psi_0, i\beta, \mu^*)$ is a Fredholm operator of index 0. Thus to verify (10), it suffices to show that the system

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_0 = \mu^*\hat{u}(s^*)G(\cdot, s^*), \qquad \hat{\lambda} = \mu^*\hat{u}(s^*) \tag{12}$$

necessarily implies that $\hat{u} = 0$, $\hat{\lambda} = 0$. Thus let $(\hat{u}, \hat{\lambda})$ be a solution of (12), and define $\psi_1 := \psi_0 - G(\cdot, s^*)$. Then,

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0. \tag{13}$$

On the other hand, since ψ_0 solves (9) with $\lambda = i\beta$ and $s = 1$, we have

$$i\beta G(\cdot, s^*) = A\psi_0 + i\beta\psi_0 = A\psi_1 + \delta_{s^*} + i\beta G(\cdot, s^*)$$

in the weak sense. Here δ_β is the delta distribution centered at s. So ψ_1 is a solution to the equation

$$(A + i\beta)\psi_1 = -\delta_{s^*}, \tag{14}$$

and

$$i\beta = \mu^*(\gamma'(s^*) + \psi_0(s^*)) = \mu^*(\gamma'(s^*) + \psi_1(s^*) + G(s^*, s^*)). \tag{15}$$

Eq. (14) implies that

$$-\overline{\psi_1(s^*)} = \int_0^1 |A^{1/2}\psi_1|^2 + i\beta\int_0^1 |\psi_1|^2,$$

so that

$$\operatorname{Im}\psi_1(s^*) = \beta\int_0^1 |\psi_1|^2.$$

Now $\gamma'(s^*)$ and $G(s^*, s^*)$ in (15) are real-valued; therefore, since $\beta \neq 0$,

$$\mu^*\int_0^1 |\psi_1|^2 = 1. \tag{16}$$

From (14) we can then calculate $\hat{u}(s^*)$ as $\int_0^1 \psi_1(A + i\beta)\hat{u} = -\hat{u}(s^*)$, which together with (13), (14) and (16) implies that

$$\hat{\lambda}\int_0^1 \psi_1^2 = \hat{u}(s^*) = \frac{\hat{\lambda}}{\mu^*} = \hat{\lambda}\int_0^1 |\psi_1|^2.$$

As a result,

$$\hat{\lambda}\left(\int_0^1 |\psi_1|^2 - \psi_1^2\right) = 0,$$

which implies $\hat{\lambda} = 0$, for otherwise $\operatorname{Im}\psi_1 = \operatorname{Im}\psi_0 = 0$, which is a contradiction. So we conclude that $\hat{\lambda} = 0$, and with this that also $\hat{u} = 0$.

We have thus shown (10), and therefore get a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata such that $\phi(\mu^*) = \phi^*$ and $\lambda(\mu^*) = i\beta$. It remains to be shown that $\operatorname{Re}\lambda'(\mu^*) \neq 0$. Let $\phi(\mu) = (\psi(\mu), 1)$. Implicit differentiation of $E(\psi(\mu), \lambda(\mu), \mu) = 0$ (see (11)) implies that

$$D_{(\mu,\lambda)}E(\psi_0, i\beta, \mu^*)(\psi'(\mu^*), \lambda'(\mu^*)) = (\gamma'(s^*) + \psi'(\mu^*)(s^*))(G(\cdot, s^*), 1).$$

This means that the function $\hat{u} := \psi'(\mu^*)$ and $\hat{\lambda} := \lambda'(\mu^*)$ satisfy the equations

$$(A + i\beta)\hat{u} - \mu^*\hat{u}(s^*)G(\cdot, s^*) + \hat{\lambda}\psi_0 = (\gamma'(s^*) + \hat{u}(s^*))G(\cdot, s^*) \tag{17}$$

and

$$\mu^*\hat{u}(s^*) + \hat{\lambda} = \gamma'(s^*) + \hat{u}(s^*). \tag{18}$$

Putting (18) into (17) and using $\psi_1 := \psi_0 - G(\cdot, s^*)$, as before, we obtain

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0,$$

and from here with (14) that

$$-\overline{\hat{u}(s^*)} = \int_0^1 (A + i\beta)\psi_1\bar{\hat{u}} = \int_0^1 \psi_1\overline{(A + i\beta)\hat{u}} = -\bar{\hat{\lambda}}\int_0^1 |\psi_1|^2 = -\bar{\hat{\lambda}}\frac{1}{\mu^*},$$

where we have used (16) for the last step. We thus get $\hat{\lambda} = \mu^*\hat{u}(s^*)$ and via (18) that

$$\hat{\lambda} = \operatorname{Re}\hat{\lambda} = -\mu^*\gamma'(s^*) \neq 0. \qquad \square$$

Remark 11. In fact (16) shows that a Hopf point necessarily has to occur where $\mu^* > 0$, the reasonable case for our model. From $\gamma'(s^*) < 0$, we conclude then that $\operatorname{Re}\lambda'(\mu^*) = -\mu^*\gamma'(s^*) > 0$, so the steady state loses stability as μ increases beyond μ^*.

As a final result we will now show that, whenever (R) admits a stationary solution, there is a unique $\mu^* > 0$ such that $(0, s^*, \mu^*)$ is a Hopf point, thus μ^* is the origin of a branch of nontrivial periodic orbits. To do this, we have only to show that the function $(u, \beta, \mu) \mapsto E(u, i\beta, \mu)$ has a unique zero with $\beta > 0$ and $\mu > 0$. This means solving the system

$$(A + i\beta)u = \mu(\gamma'(s^*) + u(s^*))G(\cdot, s^*), \qquad i\beta = \mu(\gamma'(s^*) + u(s^*)).$$

As before, with $v := u - G(\cdot, s^*)$, this system is equivalent to the weak system of equations

$$(A + i\beta)v = -\delta_{s^*}, \qquad i\beta = \mu(\gamma'(s^*) + G(s^*, s^*) + v(s^*)). \tag{19}$$

(Actually, this is just the eigenvalue problem for the formal linearization of (F) about (v^*, s^*).)

Now the first equation in (19) has, for fixed $\beta \geqslant 0$, the unique solution $v = -G_\beta(\cdot, s^*)$, where G_β is the Green's function for the operator $A + i\beta$. We are thus left with having to solve the complex-valued equation

$$i\beta = \mu\big(\gamma'(s^*) + G(s^*, s^*) - G_\beta(s^*, s^*)\big).$$

Since $\gamma'(s^*) + G(s^*, s^*)$ is real-valued, this is equivalent to the real-valued system

$$\gamma(s^*) + G(s^*, s^*) - \operatorname{Re} G_\beta(s^*, s^*) = 0, \tag{20}$$

$$\mu \operatorname{Im} G_\beta(s^*, s^*) + \beta = 0. \tag{21}$$

Since (20) does not depend on μ, it suffices to find a unique solution $\beta > 0$ of (20); from this β the unique $\mu^* > 0$ can then be calculated using (21), provided $\operatorname{Im} G_\beta(s^*, s^*)$ is negative. Thus, everything follows from the following lemma.

Lemma 12. *The expression* $\operatorname{Re} G_\beta(s^*, s^*)$ *is strictly decreasing in* $\beta \in \mathbb{R}^+$ *with*

$$\operatorname{Re} G_0(s^*, s^*) = G(s^*, s^*), \qquad \lim_{\beta \to \infty} \operatorname{Re} G_\beta(s^*, s^*) = 0,$$

and $\operatorname{Im} G_\beta(s^*, s^*) < 0$ *for any* $\beta > 0$.

Proof. First we have $(A + \mathrm{i}\beta)^{-1} = (A - \mathrm{i}\beta)(A^2 + \beta^2)^{-1}$, so if $L(\beta) := \operatorname{Re}(A + \mathrm{i}\beta)^{-1}$ and $T(\beta) := \operatorname{Im}(A + \mathrm{i}\beta)^{-1}$, then

$$L(\beta) = A(A^2 + \beta^2)^{-1} \quad \text{and} \quad T(\beta) = -\beta(A^2 + \beta^2)^{-1}.$$

Since $(A^2 + \beta^2)^{-1}$ is a positive operator, it follows that $-T(\beta)$ is positive for $\beta > 0$, which implies that $\operatorname{Im} G_\beta(s^*, s^*) < 0$. Moreover, $L(\beta) \to A^{-1}$ as $\beta \to 0$ and $L(\beta) \to 0$ as $\beta \to \infty$, which results in the corresponding limiting behavior for $\operatorname{Re} G_\beta(s^*, s^*)$.

Now to show that $\beta \mapsto G_\beta(s^*, s^*)$ is strictly decreasing, define $h(\beta)(x) := G_\beta(x, s^*) - G(x, s^*)$. Then (in the weak sense at first)

$$(A + \mathrm{i}\beta)h(\beta) = -\mathrm{i}\beta G(\cdot, s^*).$$

As a result $h(\beta) \in D(A)_{\mathbb{C}}$ and $h: \mathbb{R}^+ \to D(A)_{\mathbb{C}}$ is differentiable with $\mathrm{i}h(\beta) + (A + \mathrm{i}\beta)h'(\beta) = -\mathrm{i}G(\cdot, s^*)$; therefore,

$$(A + \mathrm{i}\beta)h'(\beta) = -\mathrm{i}G_\beta(\cdot, s^*).$$

We thus get

$$\begin{aligned} -\mathrm{i}\overline{h'(\beta)(s^*)} &= \int_0^1 (A + \mathrm{i}\beta)^2 h'(\beta)\overline{h'(\beta)(x)}\,\mathrm{d}x \\ &= \int_0^1 (A + \mathrm{i}\beta)h'(\beta)(A + \mathrm{i}\beta)\overline{h'(\beta)}\,\mathrm{d}x \\ &= \int_0^1 |Ah'(\beta)|^2 - \beta^2 |h'(\beta)|^2\,\mathrm{d}x + 2\mathrm{i}\beta \int_0^1 Ah'(\beta)\overline{h'(\beta)}\,\mathrm{d}x. \end{aligned}$$

It follows that

$$\operatorname{Re} h'(\beta)(s^*) = -2\beta \int_0^1 |A^{1/2}h'(\beta)|^2 < 0,$$

because $\operatorname{Re} h'(\beta)$ cannot vanish identically, and the lemma is proved. □

Now, since $\gamma'(s^*) < 0$ and $\gamma'(s^*) + G(s^*, s^*) > 0$, the existence of a unique solution (β, μ^*) of (20) and (21) with $\beta > 0$ and $\mu^* > 0$ follows by an application of the mean value theorem. For given values of c and a, μ^* can be obtained numerically by applying Newton's method to (20). For the symmetric situation $a = \frac{1}{4}$ and $c^2 = 2$, we have found that $\tau_c = 4/\mu^* \approx 0.27$.

The following theorem summarizes the things we have proved.

Theorem 13. *Assume that* $0 < \frac{1}{2} - a < 1/c^2$, *so that* (R), *respectively* (F), *has a unique stationary solution* $(0, s^*)$, *respectively* (v^*, s^*), *for all* $\mu > 0$. *Then there exists a unique* $\mu^* > 0$ *such that the linearization* $-\bar{A} + \mu^* B$ *has a purely imaginary pair of eigenvalues. The point* $(0, s^*, \mu^*)$ *is*

then a Hopf point for (R) *and there exists a* C^0*-curve of nontrivial periodic orbits for* (R), (F), *respectively, bifurcating from* $(0, s, \mu^*)$, (v^*, s^*, μ^*), *respectively.*

4. Numerics

In Section 3 we have shown that the problem (F) undergoes a Hopf bifurcation as $\mu \to \infty$. In this section, we will give a brief outline of the numerical algorithm used to produce the simulations of the free boundary problem given in Figs. 4 and 5. The numerical tools used are fairly elementary — a combination of Euler's method and the trapezoidal rule is used to advance v and s in the time t direction, together with a finite-difference approximation of the two-point boundary value problem for v on each fixed time line.

From a numerical perspective, the problem consists of approximating the solution to a partial differential equation for v, which we write in the form

$$v_{xx} = v_t + 2v - H(x - s), \tag{22}$$

coupled with an ordinary differential equation for s, which we write in the form

$$\frac{ds}{dt} = \frac{1}{\tau} c(v(s)), \tag{23}$$

subject to the initial conditions

$$v(x, 0) = v_0(x), \qquad s(0) = \hat{s}_0,$$

and the boundary conditions

$$v_x(0, t) = 0 = v_x(1, t). \tag{24}$$

We introduce a grid on the domain $[0, 1] \times [0, \infty)$ by setting a spatial increment Δx and a time increment Δt and defining $x_j = j\,\Delta x$ and $t_k = k\,\Delta t$. The index j runs from 0 to nx with $nx\,\Delta x = 1$. The quantities Δx and Δt remain fixed during the calculation. We denote by v_{jk} and s_k the numerical approximations to $v(x_j, t_k)$ and $s(t_k)$, respectively. Naturally $v_{j0} = v_0(x_j)$ and $s_0 = \hat{s}_0$.

Assume now that v_{jk} and s_k, which approximate v and s on the time line $t_k = k\,\Delta t$, have been calculated. We will now describe how to advance the solution to the next time line. The method proceeds in several stages, as follows.

Stage 1: predictor. An initial approximation s_{k+1}^0 to s_{k+1} is obtained by applying Euler's method to the differential equation (23) for s in the form

$$s_{k+1} = s_k + \frac{\Delta t}{\tau} c(v^{k, s_k}),$$

where v^{k, s_k} is an approximation to $v(s_k, t_k)$, obtained from the values of v_k by linear interpolation. (This interpolation is necessary because the values of s_k will not usually fall on spatial grid lines.)

Next an initial approximation $v_{j,k+1}^0$ is obtained from (22) by solving the second-order difference equation

$$\frac{v_{j+1,k+1} - 2v_{j,k+1} + 2v_{j-1,k+1}}{(\Delta x)^2} = \frac{v_{j,k+1} - v_{j,k}}{\Delta t} - 2v_{j,k+1} - H\left(x_j - s_k^0\right). \tag{25}$$

The boundary conditions (24) at $x = 0$ and $x = 1$ are approximated by

$$v_x(0, t_{k+1}) \approx \frac{v_{1,k+1} - v_{-1,k+1}}{\Delta x} = 0 \quad \text{and} \quad v_x(1, t_{k+1}) \approx \frac{v_{nx+1,k+1} - v_{nx-1,k+1}}{\Delta x} = 0.$$

Under these assumptions, the finite-difference equation (25) can be written as a matrix equation of the form

$$AV = F,$$

where V is an $(nx + 1) \times 1$-dimensional column vector

$$v = \begin{pmatrix} v_{0,k+1} \\ v_{1,k+1} \\ \vdots \\ v_{ns-1,k+1} \\ v_{ns+1,k+1} \end{pmatrix}.$$

The coefficient matrix A is a tridiagonal matrix of the form

$$A = \begin{pmatrix} b_0 & c_0 & & & \\ a_1 & b_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{ns} & b_{ns} & c_{ns} \\ & & & a_{ns+1} & b_{ns+1} \end{pmatrix},$$

with $a_i = 1$ for $i = 1, \ldots, nx - 1$, $a_{nx} = 2$, $b_i = -2 - (\Delta x)^2(2 + 1/\Delta t)$ for $i = 0, \ldots, nx$, and $c_i = 1$ for $i = 1, \ldots, nx - 1$, $c_0 = 2$, and the $nx + 1 \times 1$ vector f on the right-hand side is given by $f_i = -(v_{i+1,k+1}\ \Delta t + H(x_i - s_{k+1}))(\Delta x)^2$.

This tridiagonal system is solved by a numerical implementation of Gaussian elimination with partial pivoting.

Stage 2: corrector. In this corrector stage, an implicit scheme is used to correct the results in the predictor stage, in order to improve the accuracy of the approximation. An updated approximation s_{k+1}^1 to s_{k+1} is obtained by applying the trapezoidal rule to the differential equation (23) as follows:

$$s_{k+1} = s_k + \frac{\Delta t}{2\tau} c\left(v_{k+1,s_{k+1}^0}^0\right) + c(v_{k,s_k}),$$

where, again, linear interpolation is used to approximate the value $v_k(s_k)$,

Using this s_{k+1}^1, corrected approximations $v_{j,k+1}^1$ are obtained by solving the difference equation (25). This concludes the advance of the solution from time $t = t_k$ to $t = t_{k+1}$. The results of this correction step are stored as approximations to v and s for $t = t_{k+1}$.

In generating the figures which appear in Section 1, this algorithm was implemented with $\Delta x = 0.025$ ($nx = 399$) and $\Delta t = 0.01$. The initial conditions were given by $v_0(x) = \frac{1}{2}x$. The critical value of τ has been calculated from the equation given in Section 3, using Newton's method, to be approximately $\tau = 0.27$. The numerical procedures described above give approximate values of v and s which are accurate to the order predicted by the size of the time and spatial grids.

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